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Quantum Algorithms for Solving Dynamic Programming Problems

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We present a general quantum algorithm for solving finite-horizon dynamic programming problems. Up to polylogarithmic factors, our algorithm provides a quadratic quantum advantage in terms of the number of states of a given dynamic programming problem. This speedup comes at the expense of the appearance of other polynomial factors representative of the number of actions of the dynamic programming problem, the maximum value of the instantaneous reward, and the time horizon of the problem. Our algorithm can be applied to combinatorial optimization problems solved classically using dynamic programming techniques. As one application, we show that the travelling salesperson problem can be solved in $O^*(\lceil c \rceil^4 \sqrt{2^n})$ on a quantum computer, where n is the number of vertices of the underlying graph and [c] is its maximum edge-weight. As another example, we show that the minimum set-cover problem can be solved in $O(\sqrt{2^n} \operatorname{poly}(m,n))$, where m is the number of sets used to cover a universe of size n. Finally, we prove lower bounds for the query complexity of quantum algorithms and classical randomized algorithms for solving dynamic programming problems, and show that no greater-than-quadratic speedup in either the number of states or number of actions can be achieved for solving dynamic programming problems using quantum algorithms.

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I. Introduction

A. Dynamic programming problems

Recently, there has been increasing interest in the construction of quantum algorithms for solving optimal control problems. This includes algorithms for finite-horizon dynamic programming [MGAG16, MLM17, ABI⁺19] and infinite-horizon reinforcement learning (RL) [DCT⁺08, BC12, CLG⁺18]. Ambainis et al. [ABI⁺19], in particular, study quantum algorithms for a collection of NP-hard problems (e.g., the travelling salesperson problem and the minimum set-cover problem) for which the best classical algorithms are exponentially expensive dynamic programming solutions. Achieving a quantum advantage over classical dynamic programming algorithms has been a long-standing problem in quantum algorithms, for which [ABI⁺19] improved the time complexity of solving these problems from $O^*(2^n)$ to $O^*(1.728^n)$.¹ While these results leave it open as to whether a quadratic quantum speedup for solving dynamic programming problems (DP) is feasible, they also require exponentially large space, as they need to store a partial dynamic programming table. Achieving a quadratic quantum speedup for solving DPs has remained an open challenge, until now.

In this paper, we introduce quantum algorithms for dynamic programming that conquer this challenge. In the most general terms, a DP is defined by a finite set of *states* S and a finite set of possible *actions* (decisions) A at each state. Performing an action at a given state results in a cost or reward and a transition to a new state. The goal is therefore to find an *optimal policy* for an action of an *agent* at every state. Here, the measure of optimality is the future reward the agent collects should she pursue the action prescribed by the optimal policy. The cumulative future reward is often called the *value function*. A policy consists of the choice of a single action at every state. If these prescribed actions are independent of the point in time the agent visits a state, the policy is called *homogeneous*. As such, a homogeneous (deterministic) policy is a function $\pi: S \to A$. In this paper, a DP is defined as the problem of finding an optimal action at a given state $s_0 \in S$ and time t = 0. An algorithm that solves this problem can then iteratively be used at every subsequent state visited by the agent until a complete optimal policy for the DP has been traversed.

B. Quantum algorithms for mathematical programming

In recent works by Brandão and Svore; Brandão et al.; van Apeldoorn, Gilyén, Gribling, and de Wolf; and van Apeldoorn and Gilyén [BS17, KLL⁺19, AGGW17, AG18]; quantum Gibbs sampling has been used to achieve a quadratic speedup in solving semidefinite programming problems (SDP) and, as a special case, linear programming problems (LP). The quadratic speedup

¹ Here, the O^* notation ignores polynomial factors in n.

is in terms of the parameters that define the size of the problem (the number of variables and constraints). This speedup comes at the expense of much worse scaling in terms of the precision of the solution. For example, van Apeldoorn et al. [AGGW17] propose a quantum algorithm for LPs that requires $\tilde{O}(\varepsilon^{-5})$ quantum gates, and an algorithm for SDPs that requires $\tilde{O}(\varepsilon^{-8})$ quantum gates, where ε is an additive error on the accuracy of the final solution. Van Apeldoorn and Gilyén improve upon the scaling of their result by further analysis and reduce the dependence on precision parameters to $\tilde{O}(\varepsilon^{-4})$ in [AG18] and more recently to $\tilde{O}(\varepsilon^{-3.5})$ in [AG19]. Several lower bounds proven in [AGGW17, AG18] suggest that these results cannot be improved significantly further, in particular, in that the computational complexity found is tight with respect to the size of the SDP, and the polynomial dependence on precision parameters is inevitable. It therefore appears that there might be a regime of parameters in which if a real-world problem falls, these quantum algorithms could be of practical advantage. The mentioned references leave open the question of finding real-world applications of LPs and SDPs that fall into this regime.

In this paper, we consider the LP formulations of DPs. As will become apparent, the conditions assumed for the quantum LP and SDP solvers of the earlier work referenced above are too strong, and thus cannot be used as off-the-shelf optimizers for the LP formulation of DPs. For instance, in working with DPs, it is crucial to modify the conventional formulation and, moreover, to consider its dual. Further, the error introduced in the objective value itself can be very large and, therefore, we study the basic feasible solutions and use complementary slackness to find an optimal action given the approximation we receive for the optimal dual solution. Nevertheless, we use the meta-algorithm known as the multiplicative weights update method (MWUM) on our feasibility problem. MWUM, in turn, creates *simpler* LPs defined on a simplex. We then use the quantum minimum finding algorithm [DH96] to solve them. It is worth mentioning that [BS17, KLL⁺19, AGGW17, AG18] also use (a matrix generalization of) MWUM [AK07]. However, unlike in these references, in our method the slave problems generated by MWUM are simpler and can be solved by the quantum minimum finding algorithm rather than having to resort to using a quantum Gibbs sampler.

C. Summary of results

We consider a finite-horizon DP with a finite space of states S and a finite space of actions A. An integer $T \in \mathbb{N}$ denotes the time horizon of the problem, which is the total duration allowed for a policy to achieve maximum utility. We consider a value function

$$V = V(\pi, s) : \Pi \times S \to \mathbb{R}.$$

Here, Π denotes the space of all deterministic policies defined along the finite time horizon. For time-homogeneous policies, $\Pi = A^S$ is the space of all functions $S \to A$, and for inhomogeneous policies, $\Pi = A^{S \times \{0, \dots, T-1\}}$. We assume that a marked initial state $s_0 \in S$ is given. The goal is to find an optimal policy at s_0 , that is,

$$\operatorname{argmax}_{\pi} V(\pi, s_0)$$

It is easy to verify through Bellman's recursion [Bel13] that, if the optimal value function $V^* = V(\pi^*, -)$ is known at all states $s \in S$ accessible from s_0 , then an optimal action at s_0 can be extracted from the optimal value function. Alternatively, an optimizer of V^* , that is, the s_0 component of an optimal policy, may be directly calculated. In this paper, we use the latter

approach.

We show that for a finite-horizon dynamic programming problem with a time horizon T, there exists a quantum algorithm that finds an optimal action at a given state $s_0 \in S$ in $\tilde{O}\left(|A|^{4.5}\rho^4\sqrt{|S|T}\right)$. We can see that this scaling is not ideal in terms of the number of actions |A|, but is quadratically faster than expected in terms of the number of states |S|. In fact, for practical applications the number of possible states is often exponentially large with respect to the length of the input string defining the problem, but the number of actions is a polynomial of it. Therefore, achieving a quadratic speedup in |S| at the expense of having a poor polynomial scaling in |A| is very valuable. We verify this fact when we apply our algorithm to the travelling salesperson problem (TSP) and the minimum set-cover problem (MSC). In the TSP, the state space corresponds to all possible subsets of the vertices of the graph, paired with a marked vertex in the subset, but the number of actions is at most equal to the number of vertices in the graph. In MSC, the states correspond to subsets of the family of sets forming a cover for the universe, whereas the dynamic programming formulation requires a constant number of actions, namely, two of them, representative of the inclusion and the exclusion of a subsequent set.

The factor ρ is a representative of the precision in which the reward structure of the DP is defined. For simplicity, we assume that the instantaneous rewards are positive integers. In this scenario, ρ is the upper bound on the total reward function over the time horizon T. If the instantaneous reward is bounded above by an integer $\lceil r \rceil$, then $\rho = O(\lceil r \rceil T)$.

Our quantum algorithm makes coherent queries to the oracle

$$\ket{s}\ket{a}\ket{x}\ket{y}\mapsto \ket{s}\ket{a}\ket{x\oplus a(s)}\ket{y\oplus r(s,a)}.$$

Therefore, our algorithm achieves an $\widetilde{O}(\sqrt{S})$ gate count with respect to the number of states of the DP, provided that the oracle defining the transitions of the states under each action $a: S \to S$ has an *efficient description* (i.e., it uses polylogarithmic numbers of qubits and gates in terms of |S| and |A|). Moreover, our quantum algorithm uses a polylogarithmic number of qubits in terms of |S|, |A|, and T, and is therefore space-efficient.

As practical applications, we show how our algorithm can be applied to solving various combinatorial optimization problem. We consider the TSP and MSC as two examples.

The TSP is the problem of finding an optimal tour between vertices of a graph. We consider n vertices with a cost of c_{ij} for travelling from vertex i to vertex j, where all c_{ij} are bounded above by an integer $\lceil c \rceil > 0$. We follow the Bellman–Held–Karp [Bel61, HK62] formulation of the TSP as a DP in which the states are defined by (S, i) where i is a vertex and $S \subseteq \{1, \ldots, n\}$ is a subset of the vertices. An oracle like the one above can be constructed for this problem using $\widetilde{O}(n^2)$ registers that are prepared in c_{ij} . In Section IIID, we show that our algorithm can solve this problem using $O^*\left(\lceil c \rceil^4 \sqrt{2^n}\right)$ quantum gates, which is, to the best of our knowledge, the first quantum algorithm to provide a quadratic speedup in solving the TSP. Of course, this speedup comes at the expense of a polynomial (quartic) dependence on the maximum edge-weight $\lceil c \rceil$.

In MSC, a set called the *universe* comprises n elements, and m subsets of it are given. The task is to find the minimum number of these subsets required for covering the entire universe. In our DP formulation, an ordering of the mentioned subsets is fixed. The states of DP then correspond to various choices of a fraction of these subsets. The (only two) actions correspond to inclusion or exclusion of the subsequent subset as we proceed along the ordering. In Section IIIE, we show that there exists a quantum algorithm for MSC that can be implemented on a circuit with $O(\sqrt{2^n} \operatorname{poly}(m, n))$ quantum gates acting on O(mn) qubits.

	Quantum lower bound	Quantum upper bound	Classical lower bound
Query complexity	$\Omega(\sqrt{ S A })$	$\widetilde{O}\left(A ^{4.5}\rho^4\sqrt{ S T}\right)$	$\Omega(S A)$

TABLE I. Summary of main results

We also report on the classical and quantum query complexity lower bounds for solving DPs using the oracle defined above. We first provide the quantum query complexity lower bound of solving DPs using the generalized relational adversary method of [Amb02] (Section IV). We find a lower bound $\Omega(\sqrt{|S||A|})$ for finite-horizon problems so long as $T = \Omega(\log(|S|))$. This demonstrates that our quantum algorithm is optimal in |S| but the dependence on other factors (|A| and ρ) may perhaps be improved. In particular, we rule out the possibility of achieving exponential speedups in solving DPs.

Lastly, in Section V, we use ideas from the relational adversary method of [Amb02], but applied in the classical query complexity setting to prove lower bounds on the query complexity of classical bounded-error randomized algorithms that solve DPs. The oracles are similar to the ones considered in the quantum algorithms but queried classically:

$$(s,a) \mapsto (a(s), r(s,a))$$
.

We obtain the lower bound of $\Omega(|S||A|)$, thereby proving the existence of a polynomial separation between the quantum algorithm proposed in this paper and the best possible classical randomized algorithms for the same problem, with respect to |S|.

D. Organization

The paper is organized as follows. In Section II, we provide an introduction to the multiplicativeweights update method. In Section III, we propose and analyze our quantum algorithm for solving finite-horizon DPs. In the same section, as concrete applications, we show how our quantum algorithm achieves an almost quadratic speedup in solving the both the TSP and the minimum set-cover problem. In Section IV, we prove a lower bound on the quantum query complexity of solving DPs and, finally, in Section V, we prove a lower bound for the classical randomized algorithms accomplishing the same task. Consequently, this completes the proof that our method for solving DPs demonstrates a quantum advantage.

II. The multiplicative weights update method

We refer the reader to [Kal07] for an introduction to the multiplicative weights update method (MWUM). For convenience, we show the application of MWUM in solving linear feasibility problems.

Following [Kal07], we first present a general statement of MWUM. Given n experts and T iterations, every expert recommends a course of action. We are expected to make decisions about actions based on experts' recommendations and the cost of each action. In the early iterations of decision making, the naïve strategy is to pick an expert uniformly at random. The expected cost will be that of the "average" expert. In later iterations, we may observe that some experts clearly outperform others. We may, therefore, choose to reward those experts by increasing the probability of their being selected in following rounds. As will be made apparent in what follows, this revision of strategy is exactly the multiplicative weights update rule.

Let $p^{(t)}$ be the distribution from which we select the experts at iteration $t \leq T$. We now select expert $i \in \{1, \ldots, n\}$ according to this distribution. At this point, the costs of the actions recommended by the experts are obtained from the environment in the form of a vector $m^{(t)}$. We assume that all entries of $m^{(t)}$ are in the range [-1, 1].

The Generic Multiplicative Weights Algorithm Input: $\varepsilon \leq \frac{1}{2}$. Initialize: t = 1 and $w_i^{(t)} = 1$ for all i. For $t = 1, 2, \dots, T$

- 1. Choose an expert *i* with a probability proportional to her weight, i.e., with probability $p_i^{(t)} = w_i^{(t)} / \sum_i w_i^{(t)}$.
- 2. Obtain the *t*-th iteration cost vector $m^{(t)}$.
- 3. Update the selection weights of experts via

$$w_i^{(t+1)} = w_i^{(t)} (1 - \varepsilon m_i^{(t)})$$
 for all *i*. (1)

Theorem II.1. For every expert i, the above algorithm guarantees that, after T iterations, we have

$$\sum_{t=1}^{T} m^{(t)} \cdot p^{(t)} \le \sum_{t=1}^{T} m_i^{(t)} + \varepsilon \sum_{t=1}^{T} |m_i^{(t)}| + \frac{\ln n}{\varepsilon}.$$

We note that the left-hand side of the inequality represents the expected cost of the experts over T rounds and the right-hand side is an upper bound on the cost of the *i*-th expert.

Proof. The proof given in [Kal07, Theorem 2] goes as follows. Let $\Phi^{(t)} = \sum_i w_i^{(t)}$. As $m_i^{(t)} \in [-1, 1]$, we have

$$\begin{split} \Phi^{(t+1)} &= \sum_{i} w_{i}^{(t+1)} = \sum_{i} w_{i}^{(t)} (1 - \varepsilon m_{i}^{(t)}) \\ &= \Phi^{(t)} - \varepsilon \Phi^{(t)} \sum_{i} m_{i}^{(t)} p_{i}^{(t)} = \Phi^{(t)} (1 - \varepsilon m^{(t)} . p^{(t)}) \\ &\leq \Phi^{(t)} \exp(-\varepsilon m^{(t)} . p^{(t)}) \,. \end{split}$$

By induction,

$$\Phi^{(T+1)} \le \Phi^{(1)} \exp\left(-\varepsilon \sum_{t=1}^{T} m^{(t)} . p^{(t)}\right) = n \exp\left(-\varepsilon \sum_{t=1}^{T} m^{(t)} . p^{(t)}\right).$$

On the other hand,

$$\Phi^{(T+1)} \ge w_i^{(T+1)} = \prod_{t=1}^T (1 - m_i^{(t)}\varepsilon).$$

The result then follows by taking the logarithms of both sides of the above two inequalities and using $\ln\left(\frac{1}{1-\varepsilon}\right) \leq \varepsilon + \varepsilon^2$.

The application of MWUM in which we are interested is solving linear feasibility problems. Let \mathcal{P} be a convex set in \mathbb{R}^n , A be an $s \times n$ matrix, and $x \in \mathbb{R}^n$. We wish to check the feasibility of the convex program

$$Ax \ge b \tag{2}$$
 s.t. $x \in \mathcal{P}$.

Let $\delta > 0$ be an error parameter and, for every $i \in \{1, \ldots, s\}$, let A_i be the *i*-th row of A and b_i the *i*-th entry of b. We aim to design an algorithm which either solves the problem up to the additive error of δ , that is, finds $x \in \mathcal{P}$ such that

$$A_i x \ge b_i - \delta$$
 for all i ,

or proves that the system is infeasible. We also assume that there exists an algorithm Q which we treat as a black box that, given a probability distribution vector p on the s constraints, solves the feasibility problem

$$p^{T}Ax \ge p^{T}b \tag{3}$$
 s.t. $x \in \mathcal{P}$.

The feasibility problem (3) is a Lagrangian relaxation of (2) and, therefore, we may find it easier to solve in certain situations. In particular, a solution x^* for (2) satisfies (3) for every choice of probability distribution p. Equivalently, a probability distribution p for which (3) is infeasible is a proof that the original problem (2) is infeasible.

Let $\ell \geq 0$ be a bound on the absolute value of all slacks in (2). More precisely, we assume that whenever the black box returns a point $x \in \mathcal{P}$, then

$$A_i x - b_i \in [-\ell, \ell]$$
 for all *i*.

A slight simplification of [Kal07, Theorem 5] follows.

Theorem II.2. Let $\delta > 0$ be a given error parameter. Assume that $\ell \geq \frac{\delta}{2}$. Then, there is an algorithm which either solves the feasibility problem (2) up to an additive error of δ , or correctly concludes that the system is infeasible, and makes only $O\left(\frac{\ell^2 \log(s)}{\delta^2}\right)$ calls to a subprocedure Q, with an additional processing time of O(s) per call.

Proof. We associate an expert to each of the *s* constraints. The *i*-th cost is given by $m_i = \frac{1}{\ell}(A_i x - b_i)$, thus satisfying $-1 \le m_i \le 1$. At each iteration *t*, given a distribution $p^{(t)}$ over the experts, we run the subprocedure *Q* with $p^{(t)}$. If the subprocedure declares that there is no $x \in \mathcal{P}$ that satisfies

 $p^{(t)}Ax \ge p^{(t)}b$, then we halt. This is because $p^{(t)}$ is a proof that the problem (2) is infeasible. Otherwise, let $x^{(t)}$ be the solution returned by the subprocedure Q:

$$p^{(t)^T} A x^{(t)} \ge p^{(t)^T} b$$
.

We set the cost vector to $m^{(t)} := \frac{1}{\ell} \left(A x^{(t)} - b \right)$, resulting in a non-negative expected value for the cost in each iteration:

$$p^{(t)} \cdot m^{(t)} = \frac{1}{\ell} p^{(t)} \cdot \left(A x^{(t)} - b \right) = \frac{1}{\ell} \left(p^{(t)^T} A x - p^{(t)^T} b \right) \ge 0.$$

By Theorem II.1, after T iterations, we have

$$0 \leq \sum_{t=1}^{T} \frac{1}{\ell} \left(A_i x^{(t)} - b_i \right) + \varepsilon \sum_{t=1}^{T} \frac{1}{\ell} |A_i x^{(t)} - b_i| + \frac{\ln s}{\varepsilon}$$
$$= (1+\varepsilon) \sum_{t=1}^{T} \frac{1}{\ell} \left(A_i x^{(t)} - b_i \right) + 2\varepsilon \sum_{<0} \frac{1}{\ell} |A_i x^{(t)} - b_i| + \frac{\ln s}{\varepsilon}$$
$$\leq (1+\varepsilon) \sum_{t=1}^{T} \frac{1}{\ell} \left(A_i x^{(t)} - b_i \right) + \frac{2\varepsilon\ell}{\ell} T + \frac{\ln s}{\varepsilon}.$$

The subscript "< 0" in the above equations refers to the iterations t for which $A_i x^{(t)} - b_i < 0$. Since \mathcal{P} is convex, $\bar{x} = \frac{1}{T} \sum_{t=1}^{T} x^{(t)}$ is in \mathcal{P} and

$$0 \le (1+\varepsilon) \left(A_i \bar{x} - b_i \right) + 2\varepsilon \ell + \frac{\ell \ln(s)}{\varepsilon T}.$$

Now, setting $\varepsilon = \frac{\delta}{4\ell}$ and $T = \lceil \frac{8\ell^2 \ln(s)}{\delta^2} \rceil$, we obtain $A_i \bar{x} \ge b_i - \delta$.

Remark 1. The additional processing time of O(s) in this theorem is due to the processing step (3) in the pseudocode presented in this section. In our usage of MWUM, the multiplicative weights $w_i^{(t)}$ are calculated and recalculated coherently in each step, thereby avoiding the additional complexity of O(s).

In our usage of MWUM, the subprocedure Q is a quantum algorithm that efficiently solves the Lagrangian relaxation (3). In fact, the quantum algorithm can solve the feasibility problem only up to a given precision. Therefore, a variant of Theorem II.2 for approximate subprocedures is useful and stated as [Kal07, Theorem 7]. We call the subprocedure $Q \delta$ -approximate if it solves the feasibility problem (3) up to an additive error of δ . That is, given the probability distribution p, it either finds $x \in \mathcal{P}$ such that $p^T A x \ge p^T b - \delta$ or it declares correctly that (3) is infeasible.

Theorem II.3. Let $\delta > 0$ be a given precision parameter. Assume that $\ell \geq \frac{\delta}{3}$. Then, there is an algorithm which either solves the feasibility problem (2) up to an additive error of δ , or correctly concludes that the system is infeasible, making only $O\left(\frac{\ell^2 \log(s)}{\delta^2}\right)$ calls to a δ -approximate subprocedure Q.

Proof. The proof is similar to that of the previous theorem, but this time setting $\varepsilon = \frac{\delta}{6\ell}$ and $T = \lceil \frac{18\ell^2 \ln(s)}{\delta^2} \rceil$ following [Kal07, Theorem 7].

III. Solving dynamic programming problems

We solve the DP using MWUM. We define the value function as

$$V(\pi, s) = V_0(\pi, s) = \sum_{i=0}^{T-1} r_i(s_i, a_i),$$

and aim to find a deterministic, time-dependent policy $\pi_t : S \to A$ that maximizes V. Here, T is the time horizon of the DP and, for convenience, we use the notation $[T] := \{0, \ldots, T-1\}$. The following structure is given:

(a) S and A are finite sets, and the transition kernel or law of motion is $a_t: S \to S$;

(b) The reward function is a bounded, deterministic, possibly time-dependent function of states, actions, and time epochs, and for simplicity takes values in the set of natural numbers

$$r_t = r_t(s, a) : S \times A \to \mathbb{N}, \quad \forall t \in [T].$$

The boundedness condition allows us to define a positive integer denoted by $\lceil r \rceil > 0$ as an upper bound on reward values. Note that, without loss of generality (and by a constant shift of all rewards if needed), we assume a lower bound of 1 for the reward function.

Notice that by the above definition of the reward function, we have implicitly assumed all actions in set A are *admissible* for all states in S. For a DP in which this condition is not naturally satisfied by the model (i.e., some actions are not allowed at certain states), we may, without loss of generality, let an originally inadmissible action a at a state s map it with a null state additionally defined.

Bellman's optimality criteria for the value function states that an optimal policy $\pi_t^* : S \to A$ is associated to the (unique) optimal value function $V_t^*(s) = V_t(\pi_t^*, s)$ satisfying

$$V_t^*(s) = \max_a \left\{ r_t(s, a) + V_{t+1}^*(a(s)) \right\} \quad \forall t \in [T] \,.$$

We can write a linear program, the solution of which provides a solution to the above functional equation. The value function depends on the time epochs $t \in \{0, \ldots, T\}$ and states $s \in S$. For each value $V_t^*(s)$ of the value function, we assign a real variable $v_{s,t}$ and, for consistency, write the constants $r_t(s, a)$ as $r_{s,a,t}$. The LP formulation is as follows.

$$\min \sum_{s,t} v_{s,t}$$
s.t. $v_{s,t} \ge r_{s,a,t} + v_{a(s),t+1} \quad \forall a \in A, s \in S, t \in [T]$

$$v_{s,t} \ge 0 \quad \forall s \in S, t \in \{0, \dots, T\}$$

$$(4)$$

The following properties are easy to verify.

Proposition III.1. The above LP is feasible and attains a unique solution.

In this unique solution, $v_{s,T} = 0$ for all $s \in S$. In what follows, we remove $v_{s,T}$ from the set of variables and treat them as the constant 0 when they appear in the constraints.

Proposition III.2. All optimal values are integers bounded below by T - t at time t. The optimal objective value $\sum v_{s,t}$ is bounded below by $|S|\binom{T}{2}$.

Proposition III.3. All optimal values are integers bounded above by $(T-t)\lceil r \rceil$ at time t. The optimal objective value $\sum v_{s,t}$ is bounded above by $|S|\binom{T}{2}\lceil r \rceil$.

Remark 2. At time T, the total amount of reward that has been possibly collected by any policy is bounded above by T[r]. In fact, we will use the notation ρ for the maximum cumulative optimal reward obtained by an optimal policy, starting from the marked initial state s_0 . We note that this quantity is bounded above by T[r], so in the computational complexity results presented below, the term ρ can be replaced by T[r] when a better bound for it is not available.

A. Dual formulation

Recall that the complexities presented in Theorem II.2 and Theorem II.3 depend on the bounds on each feasibility slack. Consequently, the upper bound for the objective function of (4) presented in Proposition III.3 would become an issue if we were to attempt to solve this LP using MWUM. On the other hand, the LP formulation (4) finds the optimal value function for every initial state $s \in S$. Our trick for overcoming this issue is to instead assume a marked initial state s_0 and solve the LP to find only the optimal value function at that state. This LP automatically finds the optimal value function at all states reachable from s_0 as well, while avoiding the appearance of large slacks.

$$\begin{array}{ll} \min & v_{s_0,0} \\ \text{s.t.} & v_{s,t} \ge r_{s,a,t} + v_{a(s),t+1} \quad \forall a \in A, s \in S, t \in [T] \\ & v_{s,t} \ge 0 \quad \forall s \in S, t \in [T] \end{array}$$

$$(5)$$

To use the MWUM of Theorem II.3, we pass from (5) to the feasibility problem

$$v_{s_{0},0} = \sigma$$

$$v_{s,t} - r_{s,a,t} - v_{a(s),t+1} \ge 0 \quad \forall s \in S, a \in A, t \in [T]$$

$$v_{s,t} \ge 0 \quad \forall s \in S, t \in [T],$$
(6)

with the intention of performing a binary search on the value of σ . However, this feasibility problem is not easier to solve using a quantum algorithm. Instead, we form the linear programming dual of (5), which may be written as

$$\max \sum_{s,a,t} r_{s,a,t} \lambda_{s,a,t}$$
s.t. $1 - \sum_{a} \lambda_{s_0,a,0} \ge 0$

$$- \sum_{a} \lambda_{\underline{s},a,\underline{t}} + \sum_{\substack{s,a \\ a(s) = \underline{s}}} \lambda_{s,a,\underline{t}-1} \ge 0 \quad \forall (\underline{s},\underline{t}) \in S \times [T] \setminus \{(s_0,0)\}$$

$$\lambda_{s,a,t} \ge 0 \quad \forall a \in A, s \in S, t \in [T],$$
(DP)

wherein the dual variables are indexed by the primal constraints and thus denoted by $\lambda_{s,a,t}$.

By strong duality for linear programming, the optimal value of (DP) coincides with that of (5). So, we may perform a binary search on $\sigma \in [1, \rho]$ in pursuit of the optimal objective value of (DP). That is, for a given value of $\sigma \in [1, \rho]$, we define \mathcal{P}_{σ} as the simplex cut out in the non-negative orthant ($\lambda \geq 0$) by $\sum_{s,a,t} r_{s,a,t} \lambda_{s,a,t} = \sigma$. We therefore want to solve the feasibility problem

$$1 - \sum_{a} \lambda_{s_0,a,0} \ge 0$$

$$- \sum_{a} \lambda_{\underline{s},a,\underline{t}} + \sum_{\substack{s,a \\ a(s) = \underline{s}}} \lambda_{s,a,\underline{t}-1} \ge 0 \quad \forall (\underline{s},\underline{t}) \in S \times [T] \setminus \{(s_0,0)\}$$
(DP_{\sigma})
t. $\lambda_{s,a,t} \in \mathcal{P}_{\sigma}$.

It turns out that we can solve this feasibility problem only δ -approximately. Therefore, it is useful to also introduce the LP

$$\max \sum_{s,a,t} r_{s,a,t} \lambda_{s,a,t}$$
s.t. $1 - \sum_{a} \lambda_{s_0,a,0} \ge -\delta$

$$- \sum_{a} \lambda_{\underline{s},a,\underline{t}} + \sum_{\substack{s,a \\ a(s) = \underline{s}}} \lambda_{s,a,\underline{t}-1} \ge -\delta \quad \forall (\underline{s},\underline{t}) \in S \times [T] \setminus \{(s_0,0)\}$$

$$\lambda_{s,a,t} \ge 0 \quad \forall a \in A, s \in S, t \in \{0, \dots, T-1\},$$
(DP ^{δ})

and its feasibility variant

 \mathbf{s} .

$$1 - \sum_{a} \lambda_{s_0,a,0} \ge -\delta,$$

$$- \sum_{a} \lambda_{\underline{s},a,\underline{t}} + \sum_{\substack{s,a \\ a(s) = \underline{s}}} \lambda_{s,a,\underline{t}-1} \ge -\delta \quad \forall (\underline{s},\underline{t}) \in S \times [T] \setminus \{(s_0,0)\}$$
(DP^{\delta}_{\sigma})
i.t. $\lambda_{s,a,t} \in \mathcal{P}_{\sigma}.$

We now study some properties of this LP that will be useful in proving Theorem III.5.

1. Basic feasible solutions

 \mathbf{S}

Let the constraints in (DP) be written as a system of linear equations with the addition of the slack variables $\eta = (\eta_{\underline{s},\underline{t}}) \ge 0$ as

$$(A \mid I) \begin{pmatrix} \lambda \\ \eta \end{pmatrix} = b,$$

where $A = A_{S \times T, S \times A \times T}$ is the matrix with entries $A_{(\underline{s},\underline{t}),(s,a,t)} = \delta_{\underline{s},s}\delta_{\underline{t},t} - \delta_{\underline{s},a(s)}\delta_{\underline{t},t+1}$ and $b = b_{S \times T,1}$ is the column matrix with entries $b_{(\underline{s},\underline{t})} = \delta_{\underline{s},s_0}\delta_{\underline{t},0}$.

From the basics of linear programming, we know that the properties of the LP (e.g., optimal solutions) may be studied by considering the basic feasible solutions as follows. Let B be a choice of basis from $\tilde{A} = (A \mid I)$. The columns of B are indexed by $J \sqcup H$, where $J = \{(s_1, a_1, t_1), \ldots, (s_k, a_k, t_k)\}$ and $H = \{(\underline{s}_1, \underline{t}_1), \ldots, (\underline{s}_\ell, \underline{t}_\ell)\}$ such that $k + \ell = |S \times T|$. We let \overline{H} denote the complement of H in $S \times T$. Then, B can be partitioned as

$$B = \begin{pmatrix} B_{HJ} & B_{HH} \\ B_{\bar{H}J} & B_{\bar{H}H} \end{pmatrix} = \begin{pmatrix} B_{HJ} & I \\ B_{\bar{H}J} & 0 \end{pmatrix}.$$

B is invertible whenever $B_{\bar{H}J}$ is invertible, and the inverse is of the form

$$C = \begin{pmatrix} 0 & C_{J\bar{H}} \\ I & C_{H\bar{H}} \end{pmatrix}$$

such that $C_{J\bar{H}} = B_{\bar{H}J}^{-1}$ and $C_{H\bar{H}} = -B_{HJ}C_{J\bar{H}}$. The associated basic feasible solution is given by

$$\begin{pmatrix} \lambda_J \\ \eta_H \end{pmatrix} = B^{-1}b = \begin{pmatrix} 0 & C_{J\bar{H}} \\ I & C_{H\bar{H}} \end{pmatrix} \begin{pmatrix} b_H \\ b_{\bar{H}} \end{pmatrix} = \begin{pmatrix} B_{\bar{H}J}^{-1}b_{\bar{H}} \\ b_H - B_{HJ}B_{\bar{H}J}^{-1}b_{\bar{H}} \end{pmatrix}.$$

2. Complementary slackness

For every error vector $\delta \geq 0$, the dual of (DP^{δ}) is of the form

$$\min \quad v_{s_0,0} + \sum \delta_{s,t} v_{s,t}$$
s.t.
$$v_{s,t} \ge r_{s,a,t} + v_{a(s),t+1} \quad \forall a \in A, s \in S, t \in [T],$$

$$(7)$$

which is identical to (5) at $\delta = 0$. By Proposition III.2, all $v_{s,t}^*$ are strictly positive and therefore complementary slackness implies that, for any optimal solution of (DP^{δ}) , all constraints are binding. As a result, for every basic feasible solution that is a candidate for optimality, the set H in the notation of the previous section is empty and \bar{H} is the entire $S \times [T]$. In what follows, a basic matrix $B_{J\bar{H}}$ will be denoted simply by B_J when $\bar{H} = S \times [T]$.

On the other hand, for any optimal action $a \in A$ at state s_0 , the constraint $v_{s_0,0} \ge r_{s,a,t} + v_{a(s_0),1}$ is binding. Therefore, $\lambda_{s_0,a,0}^*$ can only be nonzero for optimal $a \in A$. The fact that $1 - \sum_a \lambda_{s_0,a,0} \ge -\delta$ is also binding implies that not all $\lambda_{s_0,a,0}^*$ can be zero simultaneously. In fact, for any choice of $\delta \ge 0$, any optimal solution λ^* attains at least one component $(s_0, a, 0)$ at which

$$\lambda_{s_0,a,0}^* \ge \frac{1}{|A|}.$$

We conclude that an optimal $a \in A$ at s_0 can be found by finding any $a \in A$ for which $\lambda_{s_0,a,0} \neq 0$. Moreover, for a $\frac{1}{2|A|}$ -approximation of λ^* , it suffices to find an $a \in A$ such that $\lambda_{s_0,a,0} \geq \frac{1}{2|A|}$.

B. Applying the multiplicative weights update method

Following Section II, we form the Lagrangian relaxation of (DP_{σ}) given a choice of Lagrange multipliers $w = (w_{s,t})$:

$$w_{s_{0},0}\left(1-\sum_{a}\lambda_{s_{0},a,0}\right)+\sum_{\underline{s},\underline{t}}w_{\underline{s},\underline{t}}\left(-\sum_{a}\lambda_{\underline{s},a,\underline{t}}+\sum_{\substack{s,a\\a(s)=\underline{s}}}\lambda_{s,a,\underline{t}-1}\right)\geq -\delta \qquad (\mathbf{L}_{\sigma,w}^{\delta})$$

s.t. $\lambda_{s,a,t}\in\mathcal{P}_{\sigma}$.

To find a feasible solution to this problem, it suffices to show that the optimal value of

$$\max \quad w_{s_0,0} \left(1 - \sum_{a} \lambda_{s_0,a,0} \right) + \sum_{\underline{s},\underline{t}} w_{\underline{s},\underline{t}} \left(-\sum_{a} \lambda_{\underline{s},a,\underline{t}} + \sum_{\substack{s,a \\ a(s) = \underline{s}}} \lambda_{s,a,\underline{t}-1} \right)$$
(8)
s.t. $\lambda_{s,a,t} \in \mathcal{P}$

is δ -approximately positive. By the fundamental theorem of linear programming, we need to check only the extreme points of the simplex $\sum_{s,a,t} r_{s,a,t} \lambda_{s,a,t} = \sigma$ to find a maximizer. These points are of the form $(0, \ldots, \sigma/r_{s,a,t}, \ldots, 0)$ for a choice of tuple (s, a, t). So, consider the function

$$\begin{split} f_{\sigma,w} : (\bar{s}, \bar{a}, \bar{t}) &\mapsto w_{s_0,0} \left(1 - \sum_a \frac{\sigma \bar{\delta}_{s_0,a,0}}{r_{s_0,a,0}} \right) + \sum_{\underline{s},\underline{t}} w_{\underline{s},\underline{t}} \left(-\sum_a \frac{\sigma \bar{\delta}_{\underline{s},a,\underline{t}}}{r_{\underline{s},a,\underline{t}}} + \sum_{\substack{s,a \\ a(s) = \underline{s}}} \frac{\sigma \bar{\delta}_{s,a,\underline{t}-1}}{r_{s,a,\underline{t}-1}} \right) \\ &= w_{s_0,0} - \sigma w_{s_0,0} \frac{\delta_{\bar{s},s_0} \delta_{\bar{t},0}}{r_{s_0,\bar{a},0}} - \sigma w_{\bar{s},\bar{t}} \frac{1}{r_{\bar{s},\bar{a},\bar{t}}} + \sigma w_{\bar{a}}(\bar{s}), \bar{t}+1} \frac{1}{r_{\bar{s},\bar{a},\bar{t}}} \\ &= w_{s_0,0} \left(1 - \sigma \frac{\delta_{\bar{s},s_0} \delta_{\bar{t},0}}{r_{s_0,\bar{a},0}} \right) + \frac{\sigma}{r_{\bar{s},\bar{a},\bar{t}}} \left(-w_{\bar{s},\bar{t}} + w_{\bar{a}}(\bar{s}), \bar{t}+1 \right) \,, \end{split}$$

where the term $w_{\bar{a}(\bar{s}),\bar{t}+1}$ contributes only when $\bar{t} < T-1$ and we have introduced $\bar{\delta}_{x,y,z} := \delta_{\bar{x},x}\delta_{\bar{y},y}\delta_{\bar{z},z}$.

With access to a quantum oracle for the above function, we can solve (8) using the quantum minimum finding algorithm (QMF). If the maximum we find is negative (with more than a determined additive error of δ), we halt. Otherwise, we continue with the update rule (1) of MWUM.

Proposition III.4 (QMF). Let $U_{\sigma,w}^{\delta}$ be a quantum circuit that acts on q qubits and computes $f_{\sigma,w}$ up to an additive error of $\frac{\delta}{2} > 0$ in its binary representation,

$$U_{\sigma,w}^{\delta}: |s\rangle |a\rangle |t\rangle |x\rangle \mapsto |s\rangle |a\rangle |t\rangle |x \oplus f_{\sigma,w}(s,a,t)\rangle.$$

There exists a quantum algorithm that, with $O(\log(1/p)\sqrt{|S||A|T})$ applications of $U_{\sigma,w}^{\delta}$ and $U_{\sigma,w}^{\delta}^{\dagger}$ and a q-multiple order of other gates, either obtains a feasible solution to $(L_{\sigma,w}^{\delta})$ up to an additive error of δ with a success probability of at least 1 - p or correctly declares it infeasible. *Proof.* We use the generalized minimum finding algorithm [AGGW17, Appendix C, Theorem 49]. If the solution returned by the quantum algorithm evaluates $f_{\sigma,w}$ to at least $-\frac{\delta}{2}$, we accept the solution as δ -approximately feasible, and declare $(L^{\delta}_{\sigma,w})$ infeasible otherwise.

Remark 3. The oracle $U_{\sigma,w}^{\delta}$ uses a register of size $O(\log(\lceil f_{\sigma,w} \rceil/\delta))$ to represent $f_{\sigma,w}$ with a precision of δ . So, the quantity q in the statement above is at least in $O(\log(\lceil f_{\sigma,w} \rceil/\delta))$.

Recall the MWUM of Theorem II.3 for an approximation oracle. The following theorem is the main technical result of this paper.

Theorem III.5. Suppose that all iterations of QMF (as described in Proposition III.4) succeed. Then, MWUM successfully solves the DP (5) in $O(|A|^2 \rho^2 \log(|S||A|T) \log(\rho))$ iterations of QMF.

Proof. We perform a binary search on $\sigma \in [1, \rho]$ in $O(\log(\rho))$ iterations. For each choice of σ , we should solve $(L^{\delta}_{\sigma,w})$ with a precision of δ to be determined. Let σ^* denote the optimal value of (DP) obtained at the optimal solution $\lambda^* = (\lambda^*_{s,a,t})$. Let $\bar{\sigma}$ be the largest value found through the binary search for which $(DP^{\delta}_{\bar{\sigma}})$ is feasible with a feasible solution $\bar{\lambda} = (\bar{\lambda}_{s,a,t})$. For the same approximation parameter $\delta > 0$, let $\tilde{\sigma}$ be the optimal value of (DP^{δ}) . By strong duality, σ^* in an integer. So, a binary search of $O(\log(\rho))$ steps guarantees that $\sigma^* \leq \bar{\sigma}$. It is also obvious that $\bar{\sigma} \leq \tilde{\sigma}$. It is easy to verify that, for any $\bar{\lambda}$ as above, either $\bar{\lambda}_{s_0,a,0} = O(1 + \delta)$ or there exists a direction along which the value of $\bar{\lambda}_{s_0,a,0}$ can be increased in the feasible region. This is sufficient to reduce the problem to showing that

$$\min_{\lambda^*} \max_{a} \left\| \tilde{\lambda}_{s_0,a,0} - \lambda^*_{s_0,a,0} \right\|,$$

where λ^* ranges over the optimizers of (DP). By the convexity of the optimal facet of the LP, this value is in turn bounded by the distance of the $(s_0, a, 0)$ -th component of basic feasible solutions that involve the index $(s_0, a, 0)$ for (DP) and (DP^{δ}) .

We conclude that, for every choice of basis, the change in the (s, a, t)-th component an optimal value of λ is in $O\left(\delta \max_J \|B_J^{-1}\|_{\infty}\right)$, where $J \subset S \times A \times [T]$ is a basis and B_J is the associated matrix of constraints as per the notation of Section IIIA1. The matrix norm is the operator norm induced by the infinity norm on euclidean vector spaces and thus coincides with the maximum absolute row sum of the representative matrix. For every choice of J, the matrix B_J may be viewed as a linear transformation $f : \mathbb{R}^J \to \mathbb{R}^{S \times [T]}$ defined in terms of elementary column vectors of real vector spaces by

$$e_{(s,a,t)} \mapsto e_{(s,t)} - e_{(a(s),t+1)} \quad \forall t \le T-2 \quad \text{and} \quad e_{(s,a,T-1)} \mapsto e_{(s,T-1)}.$$

Let G be a graph with vertices $V(G) = S \times [T]$ and edges between (s,t) and (a(s),t+1) for every $(s,a,t) \in J$. In order for B_J to form a basis, f has to be full rank. Consequently, this means that the graph G is a tree. Thus, for every (s,t) in G, there exists a unique path connecting the vertex (s,t) to a vertex (s',T-1):

$$(s,t) \xrightarrow{(s,a_1,t)} (s_1,t+1) \xrightarrow{(s_1,a_2,t+1)} \cdots \xrightarrow{(s_{T-2-t},a_{T-1-t},T-2)} (s_{T-1-t},T-1).$$

It can now easily be verified that

$$e_{(s,a_1,t)} + e_{(s_1,a_2,t+1)} + \dots + e_{(s_{T-2-t},a_{T-1-t},T-2)} + f^{-1}(e_{s,T-1})$$

is the preimage of (s,t). Therefore, for any choice of index set J, B_J is invertible only if an index $(s_0, a, 0)$ is contained in J for only a single action $a \in A$, and, for any such choice of J and error vector δ , the change of the optimal $\lambda_{(s_0,a,0)}$ is in $O(\delta)$. This, together with the complementary slackness argument of Section III A 2, implies that it suffices to set $\delta = \frac{1}{2|A|}$ and perform MWUM to find a δ -approximate solution as described in Theorem II.3 and to use QMF as a δ -approximate oracle for MWUM. Using the notation of the same theorem, we have to calculate ℓ , the upper bound on slacks in (DP_{σ}) . In the simplex $\mathcal{P}_{\sigma} : (\sum_{s,a,t} r_{s,a,t} \lambda_{s,a,t} = \sigma)$, we have $\sum_{s,a,t} |\lambda_{s,a,t}| \leq \sigma$. Therefore, each slack in (DP_{σ}) is bounded by $2\sigma \leq 2\rho$. The number of variables is |S||A|T. This all amounts to $O(|A|^2\rho^2 \log(|S||A|T) \log(\rho))$ iterations.

As expected, QMF has a nonzero probability of failure. However, we can now set this success probability sufficiently high so that with a high probability all QMF runs succeed throughout MWUM.

Corollary 1. Let $U_{\sigma,w}^{\delta}$ be a quantum circuit that acts on q qubits and computes $f_{\sigma,w}$ up to an additive error of $\delta > 0$ in its binary representation. Then, MWUM successfully solves the DP (5) with a probability of at least $\frac{1}{2}$ in

$$O\left(|A|^{2.5}\rho^2\sqrt{|S|T}\operatorname{polylog}(\rho,|S||A|T)\right)$$
(9)

calls to oracles of QMF and a q-multiple of it as the order of other gates.

Proof. If the probability of failure of a single iteration of QMF is p, we can perform a sequence of $\frac{1}{2p}$ QMF rounds with a success probability of at least $\frac{1}{2}$. By Theorem III.5, we have to perform $O(|A|^2\rho^2 \log(|S||A|T) \log(\rho))$ iterations of QMF. So, if $1/p = O(|A|^2\rho^2 \log(|S||A|T) \log(\rho))$, the entire process succeeds with no failures with a probability of at least $\frac{1}{2}$. Each round of QMF performs $O(\log(1/p)\sqrt{|S||A|T})$ calls to its oracles. In total, this amounts to (9) calls to oracles of different rounds of QMF.

C. Construction of the oracles

For a given choice of $\sigma \in [1, \rho]$ and by Theorem III.5, we have to solve problems of the form $(L_{\sigma,w}^{\delta})$. We make queries to the following oracle and its conjugate:

$$U_{\sigma,w}^{\delta}: |s\rangle |a\rangle |t\rangle |x\rangle \mapsto |s\rangle |a\rangle |t\rangle |x \oplus f_{\sigma,w}(s,a,t)\rangle,$$

where

$$f_{\sigma,w}(\bar{s},\bar{a},\bar{t}) = w_{s_0,0} \left(1 - \sigma \frac{\delta_{\bar{s},s_0} \delta_{\bar{t},0}}{r_{s_0,\bar{a},0}} \right) + \frac{\sigma}{r_{\bar{s},\bar{a},\bar{t}}} \left(-w_{\bar{s},\bar{t}} + w_{\bar{a}(\bar{s}),\bar{t}+1} \right).$$

At the k-th iteration of MWUM, the Lagrange multipliers $\omega_{s,t}$ are computed via

$$w_{s,t}^{(k)} = (1 - \varepsilon m_{s,t}^{(1)}) \cdots (1 - \varepsilon m_{s,t}^{(k-1)}),$$

where, for all choices of $s \in S$, $a \in A$, and $k \in \{1, \ldots, t\}$,

$$m_{\underline{s},\underline{t}}^{(k)} = \begin{cases} 1 - \sum_{a} \lambda_{s_0,a,0}^{(k)} & \underline{s} = s_0, \underline{t} = 0\\ -\sum_{a} \lambda_{\underline{s},a,\underline{t}}^{(k)} + \sum_{\substack{s,a \\ a(s) = \underline{s}}} \lambda_{s,a,\underline{t}-1}^{(k)} & \text{otherwise.} \end{cases}$$

Here, $\lambda_{s,a,t}^{(k)}$ is only nonzero if at the k-th iteration the simplex vertex $(s^{(k)}, a^{(k)}, t^{(k)})$ was chosen by QMF. In the case where they are nonzero, the values are of the form $\sigma^{(k)}/r_{s^{(k)},a^{(k)},t^{(k)}}$, where $\sigma^{(k)}$ is the k-th chosen σ in the binary search:

$$m_{\underline{s},\underline{t}}^{(k)} = \begin{cases} 1 - \lambda_{s_0,a^{(k)},0}^{(k)} \delta_{s^{(k)},s_0} \delta_{t^{(k)},0} & \underline{s} = s_0, \underline{t} = 0\\ \\ -\lambda_{\underline{s},a^{(k)},\underline{t}}^{(k)} + \lambda_{s^{(k)},a^{(k)},\underline{t}-1}^{(k)} \delta_{a^{(k)}(s^{(k)}),\underline{s}} & \text{otherwise.} \end{cases}$$

This can all be implemented using a bounded-size quantum circuit, with a bounded number of registers each with a number of qubits bounded by $\log(\lceil f_{\sigma,w} \rceil) = O(\operatorname{polylog}(|S||A|T, \rho))$.

We are ready to state the total gate complexity of solving the finite-horizon DP problem.

Theorem III.6. Let U be a quantum circuit that acts on q qubits and implements the transition kernel

$$|s\rangle |a\rangle |t\rangle |x\rangle |y\rangle \mapsto |s\rangle |a\rangle |t\rangle |x \oplus a(s)\rangle |y \oplus r_t(s,a)\rangle$$

of a dynamic programming problem defined on a state space S, an action space A, and a finite time horizon $T \in \mathbb{N}$. There exists a quantum algorithm that, given any state $s_0 \in S$, finds an optimal action $a \in A$ with high probability using

$$O\left(|A|^{4.5}\rho^4\sqrt{|S|T}\operatorname{polylog}(|S||A|T,\rho)\right)$$

queries to U and a q-multiple of it as the order of other gates.

Proof. By Theorem III.5, we have to solve $O(\rho^2 |A|^2 \operatorname{polylog}(|S||A|T, \rho))$ problems of the form $(\mathcal{L}_{\sigma,w}^{\delta})$. Then, the number of gates needed to construct the oracle of $w_{s,t}^{(k)}$ is also in the same order. This amounts to $O(\rho^4 |A|^4 \operatorname{polylog}(|S||A|T, \rho))$ iterations of QMF, thus completing the proof. \Box

Remark 4. The above-mentioned complexity is bounded in terms of T and the maximum instantaneous reward of the DP by

$$O\left(|A|^{4.5}T^{4.5}\lceil r\rceil^4\sqrt{|S|}\operatorname{polylog}(|S|,|A|,T,\lceil r\rceil)\right).$$

Corollary 2. Iterative application of the algorithm of Theorem III.5 finds an optimal policy for the DP with high probability in $O\left(|A|^{4.5}T^{1.5}\rho^4\sqrt{|S|}\operatorname{polylog}(|S||A|T,\rho)\right)$.

D. Example: Travelling salesperson problem

In certain combinatorial optimization problems, every state $s \in S$ is reachable only at a unique time $t \in [T]$. In these scenarios, it is useful to instead consider the pair (s, t) as the new definition of a state.

Remark 5. Consider a DP in which the state space is partitioned as $S = \bigsqcup_{t=0}^{T} S_t$ such that the states in S_t are accessible only at time t. This simplifies our linear programming formulation (5) to

$$\begin{array}{ll} \min & v_{s_0} \\ \text{s.t.} & v_s \ge r_{s,a} + v_{a(s)} \quad \forall \, a \in A, s \in S \setminus S_T \\ & v_s \ge 0 \quad \forall \, s \in S \,. \end{array}$$

$$(10)$$

Similarly, (DP) simplifies to

$$\max \sum_{s \in S \setminus S_{f}, a \in A} r_{s,a} \lambda_{s,a}$$

s.t.
$$1 - \sum_{a} \lambda_{s_{0},a} \ge 0$$
$$- \sum_{a} \lambda_{\underline{s},a} + \sum_{\substack{s,a \\ a(s) = \underline{s}}} \lambda_{s,a} \ge 0 \quad \forall \underline{s} \in S \setminus S_{T}$$
$$\lambda_{s,a} \ge 0 \quad \forall a \in A, s \in S \setminus S_{T}.$$
$$(11)$$

It is straightforward now to check that the result of Theorem III.6 implies the existence of a quantum algorithm that with high probability finds an optimal action at any input state s_0 using

$$O\left(|A|^{4.5}\rho^4\sqrt{|S|}\operatorname{polylog}(|S||A|,\rho)\right)$$

queries to a quantum circuit consisting of q qubits and a q-multiple of it as the order of other gates. The queried quantum circuit implements

$$|s\rangle |a\rangle |x\rangle |y\rangle \mapsto |s\rangle |a\rangle |x \oplus a(s)\rangle |y \oplus r(s,a)\rangle .$$

An iterative application of this quantum algorithm finds an optimal policy for the dynamic programming problem with initial state s_0 in

$$O\left(T|A|^{4.5}\rho^4\sqrt{|S|}\operatorname{polylog}(|S||A|,\rho)\right)$$

queries to the same oracle.

We now consider the concrete example of the travelling salesperson problem (TSP). Let G be a fully connected graph with n = |V| vertices. We use the integer indices $V = \{1, \ldots, n\}$ to represent these vertices. We let 1 be a fixed starting vertex and c_{ij} be the cost of travelling from vertex i to vertex j. The goal is to find a Hamiltonian cycle (a cycle that visits each vertex of the graph

exactly once) starting and ending at 1, incurring the lowest total cost. The best known classical algorithms for the TSP have a runtime of $O(n^2 2^n)$ for a graph of size n [Bel61, HK62].

We define a state to be a pair (H, i), where $i \in H$ and $H \subseteq V$. An action at a state (H, i) corresponds to the choosing of a vertex $j \in H \setminus \{i\}$. The instantaneous cost of going from state (H, i) to $(H \setminus \{i\}, j)$ is the cost of travelling from vertex j to i, that is, c_{ji} . The cost function C(H, i) represents the minimum total cost of a Hamiltonian path starting at 1, entering H immediately, traversing H, and ending in i. Bellman's optimality criterion may now be written as

$$C(H,i) = \min_{j \in H \setminus \{i\}} \left[C(H \setminus \{i\}, j) + c_{ji} \right].$$

We begin by assuming an oracle U_G for the adjacency of the weighted graph G:

$$|i\rangle |j\rangle |x\rangle \mapsto |i\rangle |j\rangle |x \oplus c_{ji}\rangle$$

We assume that the cost function c is integer-valued and bounded between 0 and an integer $\lceil c \rceil > 0$. The registers in U_G are of size $2\log(n) + \log(\lceil c \rceil)$. By preparing $O(n^2)$ registers in the values c_{ij} , we obtain an implementation of the oracle U_G using $O(n^2 \operatorname{polylog}(n, \lceil c \rceil))$ qubits.

It is trivial to move from a cost-minimizing statement to a reward-maximizing one by assigning $r_{ij} = \lceil c \rceil + 1 - c_{ij}$. The definition of states (H, i) can be extended to allow cases in which $i \notin H$ and the definition of an action j on a state (H, i) can be extended to allow $j \notin H$ by defining an instantaneous reward of 0 in both cases. We may now rewrite the dynamic programming problem as

$$V^*(H,i) = \max_{j \in H \setminus \{i\}} \Big[V^*(H \setminus \{i\}, j) + r_{ji} \Big].$$

From U_G we can construct an oracle U similar to that of Theorem III.6.

 $|H,i\rangle |j\rangle |x\rangle |y\rangle \mapsto |H,i\rangle |j\rangle |x \oplus [H \setminus \{i\},j]\rangle |y \oplus r_{ij}\rangle$

Every state $|H,i\rangle = |H\rangle |i\rangle$ is encoded using a binary string of size n that represents the subset $H \subseteq V$ and an index i encoded with $\log(n)$ qubits and stored as $|i\rangle$. Therefore, the registers in U are made from $O(n \operatorname{polylog}(n, \lceil c \rceil))$ qubits. The circuit U queries U_G and thus uses a total of $O(n^2 \operatorname{polylog}(n, \lceil c \rceil))$ qubits.

Corollary 3. Let G be a weighted directed graph of size n with edge weights in the range $\{0, \ldots, \lceil c \rceil\}$. There exists a quantum algorithm that solves the travelling salesperson problem on G starting and ending at a marked vertex v_0 in $O^*(\sqrt{2^n} \lceil c \rceil^4)$ gates and using $O(n^2 \operatorname{polylog}(n, \lceil c \rceil))$ qubits.

Proof. We are interested in the optimal value function at an initial state (V, 1). The time horizon needed for finding a Hamiltonian cycle is T = n. The optimal value $V^*(V, 1)$ is therefore in $O(n\lceil c\rceil)$. We also note that a state (H, i) is accessible only at time n - |H|. Finally, the actions for this dynamic programming formulation of the TSP correspond to the choice of vertices and therefore |A| = O(n). We conclude, using Theorem III.6 and Remark 5, that our algorithm requires $O\left(\sqrt{2^n}n^{10}\lceil c\rceil^4$ polylog $(n^22^n, n\lceil c\rceil)\right)$ queries to U. The number of queries to U_G is also in the same order. This completes the proof.

E. Example: Minimum cover-set problem

Consider set U, called the *universe*, with n elements. We denote by a family \mathcal{F} of m subsets $V \subseteq U$. The minimum cover-set problem (MCS) is the problem of finding the minimum cardinality of a subset $\mathcal{F}' \subseteq \mathcal{F}$ such that

$$\bigcup_{x\in\mathcal{F}'}x=U\,.$$

We define a DP as follows. Let the elements of \mathcal{F} be denoted by x_1, \ldots, x_m . A state s_t at a time t corresponds to the choice of a minimum cover binary choice of either the inclusion or exclusion of the first t elements x_1, \ldots, x_t . Therefore, s_t is the tuple (t, S_t, U_t) , where

$$S_t \subseteq \{x_0, \ldots, x_t\}$$

and

$$U_t = U \setminus \left(\bigcup_{x \in S_t} x\right).$$

The set of actions consists of only two elements $A = \{a_0, a_1\}$, where the transition from s_t via a_0 is to a state at time t + 1 that excludes x_t ; hence,

$$a_0(s_t) = (t+1, S_t, U_t),$$

and the transition via a_1 is to a state at time t + 1 that includes x_t . Hence,

$$a_1(s_t) = (t+1, S_t \sqcup \{x_t\}, U_t \setminus x_t)$$

The transition via a_0 occurs at no additional cost, whereas transition via a_1 adds a new set to the candidate cover set. To remain in a reward-maximizing framework, we therefore define the reward for transition with a_0 as +1 and the reward for transition with a_1 as 0. Every state at time t = m is of the form (m, S_m) , where $S_m \subseteq \mathcal{F}$, and S_m is a set cover whenever $U \setminus \bigcup_{x \in S_m} x$ is empty. We now consider a final action from any state (m, S_m) to a terminus state s_{m+1} with a reward of m+1 if and only if S_m is a set cover, and a reward of 0 otherwise.

We define a single state $s_0 = (0, \emptyset, U)$ at time t = 0. It is obvious that the value function at s_0 is maximized by a policy that excludes the greater sets in \mathcal{F} and the included sets form a set cover.

Corollary 4. There exists a quantum algorithm that solves the minimum set-cover problem in $O(\sqrt{2^n} \operatorname{poly}(m, n))$ gates acting on a circuit consisting of O(mn) qubits.

Proof. Our quantum algorithm takes advantage from queries to an oracle of the form in Theorem III.6. Storing every state $|s_t\rangle$ requires O(m) qubits. Storing \mathcal{F} requires m registers, each of size O(n). These registers can be used to evaluate the immediate rewards via O(mn) gates. We conclude that the oracle of Theorem III.6 can be constructed with O(mn) qubits and O(mn) gates. In the above dynamic programming formulation of the MSC, the number of actions is O(1). The time horizon of the DP is O(m). The number of states is $O(m2^n)$ and the value function is bounded by O(m). By Remark 5, the algorithm makes

$$O\left(m^{5.5}\sqrt{2^n}\operatorname{poly}(n)\operatorname{polylog}(m)\right)$$

queries to the oracle of the MSC and uses an O(mn) multiple of the number of queries as the order of other gates. The number of gates used in the construction of the oracle itself is O(mn). This amounts to a total of

$$O\left(m^{7.5}\sqrt{2^n}\operatorname{poly}(n)\operatorname{polylog}(m)\right)$$

quantum gates for the entire algorithm.

IV. Quantum complexity lower bound

We now investigate the quantum query complexity of solving DPs using the adversary method of [Amb02]. Consider two families of DP problem instances \mathcal{M}_1 and \mathcal{M}_2 sharing the same state space $S = S_0 \sqcup S_1 \sqcup S_B \sqcup S_G$, the same action space A, and the same time horizon $T \in \mathbb{N}$. We assume there exist positive integers m and n such that $|S_1| = |S_B| = n$ and $|A| = \frac{m}{n} > 2$. The set S_G is a singleton $|S_G| = 1$. We will assume that the elements in S_1 are ordered according to an indexing $\{1, 2, \ldots, n\}$. For all instances in \mathcal{M}_1 and \mathcal{M}_2 , every action maps $s \in S_G$ to itself with a reward of 1 and every $s \in S_B$ to itself with a reward of 0.

The structure of S_0 is also common between DP problem instances in \mathcal{M}_1 and \mathcal{M}_2 . The initial starting point for all instances is a marked state $s_0 \in S_0$. The role of S_0 is to make every state in S_1 accessible from s_0 in $\lceil \log n \rceil$ steps. Let $a_L, a_R \in A$ be two fixed actions. The states in S_0 form a binary tree with s_0 as the root. Actions a_L and a_R map every parent state to its left and right children (which might coincide) with a reward of 0, and every action $a \in A \setminus \{a_L, a_R\}$ maps every state in S_0 to itself with a reward of 1. The leaves of the tree consist of $b_1 = \lceil |S_1|/2 \rceil$ states. The leaves have $b_2 = \lceil b_1/2 \rceil$ parents, and so on. Since there exists a power of two between n and 2n, for all i we have $b_i \leq n/2^{i-1}$. Therefore, $|S_0| \leq 2n$ and, consequently, |S| = O(n).

For any $M_1 \in \mathcal{M}_1$, every $a \in A$ maps every $s \in S_1$ to some $a(s) \in S_B$ with a reward of 0. Therefore, the optimal value function for M_1 at s_0 is $v_{M_1}^*(s_0) = T$ and any action $a \neq a_L, a_R$ is optimal. The instances $M_2 \in \mathcal{M}_2$ differ from those in \mathcal{M}_1 only in a special state–action pair $(\bar{s}, \bar{a}) \in S_1 \times A$ for which $\bar{a}(\bar{s})$ is the single element of S_G with a reward of $r_1 = T$. So long as $T \geq \lceil \log(n) \rceil$, we have $v_{M_2}^*(s_0) > T$ and the optimal action at s_0 is one of a_L or a_R , depending on the choice of \bar{s} .

Now, consider a function $f : \{0, 1\}^* \to \{0, 1\}$ that receives a binary string describing the transition kernel of a problem instance in $\mathcal{M}_1 \sqcup \mathcal{M}_2$ and returns 0 if and only if the optimal action at s_0 is in $\{a_L, a_R\}$.

Theorem IV.1. Any quantum algorithm that computes the function f above uses $\Omega(\sqrt{|S||A|})$ queries.

Proof. We consider the relation R between instances $M_1 \in \mathcal{M}_1$ and $M_2 \in \mathcal{M}_2$ to be defined as $(M_1, M_2) \in R$ if and only if their transition kernel differs in exactly a single pair (\bar{s}, \bar{a}) . We now



FIG. 1. Schematics of instances in \mathcal{M}_1 and \mathcal{M}_2 . A pair $M_1 \in \mathcal{M}_1$ and $M_2 \in \mathcal{M}_2$ is depicted that is in relation R, as their transition kernels differ in a single state–action pair $(\bar{s}, \bar{a}) \in S_1 \times A$.

use [Amb02, Theorem 2]. We note that

- Each instance in \mathcal{M}_1 is in relation R with $|S_1||A|$ instances in \mathcal{M}_2 .
- Each instance in \mathcal{M}_2 is in relation R with $|S_B|$ instances in \mathcal{M}_1 .
- For every instance in \mathcal{M}_1 and every pair $(s, a) \in S \times A$ there is at most 1 instance in \mathcal{M}_2 with a different transition kernel $(s, a) \mapsto (a(s), r(s, a))$.
- For every instance in \mathcal{M}_2 and every pair $(s, a) \in S \times A$ there are at most $|S_B|$ instances in \mathcal{M}_1 with a different transition kernel $(s, a) \mapsto (a(s), r(s, a))$.

Then, [Amb02, Theorem 2] implies that the number of queries made by the quantum algorithm is lower bounded by

$$\Omega\left(\sqrt{\frac{|S_1||A||S_B|}{|S_B|}}\right) = \Omega\left(\sqrt{|S_1||A|}\right) = \Omega(\sqrt{|S||A|}),$$

proving the theorem.

V. Classical complexity lower bound

We now investigate the computational complexity of solving DPs classically and prove a query complexity separation between the lower bound on the classical query complexity of boundederror randomized algorithms and the upper bounds on the quantum query complexity proven in Section IV.

Once again, we borrow techniques from adversarial methods [Aar06, Amb02, CW17], but this time apply them to classical randomized algorithms. As in Section IV, we define families of DP instances \mathcal{M}_1 and \mathcal{M}_2 sharing the same state and action spaces. We then show that, if a randomized

algorithm solves DPs with high probability, there should be a deterministic algorithm μ that also succeeds in distinguishing a large fraction of the instances in the two families.

The family $\mathcal{M} = \mathcal{M}_1 \sqcup \mathcal{M}_2$ of DP instances is defined as in Section IV and Fig. 1. For an instance $M_1 \in \mathcal{M}_1$, every $a \in A$ maps every $s \in S_1$ to some $a(s) \in S_B$ with a reward of 0, except at a special state-action pair $(\bar{s}, \bar{a}) \in S_1 \times A$ at which the reward is $r_0 = 1$. The instances $M_2 \in \mathcal{M}_2$ differ from those in \mathcal{M}_1 only in this special state-action pair (\bar{s}, \bar{a}) for which $\bar{a}(\bar{s}) \in S_G$ with a reward of $r_2 = T$. By a similar argument to that in the previous section, it is obvious that an algorithm that finds an optimal action at s_0 is able to distinguish instances between \mathcal{M}_1 and \mathcal{M}_2 . It is straightforward to see that

$$|\mathcal{M}_1| = |\mathcal{M}_2| = mn^m.$$

Let Π_Q be the set of all the deterministic algorithms which, for an instance $M \in \mathcal{M}$, make at most Q queries to the oracle

$$(s,a) \mapsto \left(a(s), r(s,a)\right)$$

before returning an optimal action at s_0 . A randomized algorithm running at most Q steps is a distribution μ on Π_Q . Let $\mathcal{P}(\Pi_Q)$ be the set of all probability measures on Π_Q and a_M^{μ} be the action returned by μ on input M. Suppose there exists a randomized algorithm $\mu \in \mathcal{P}(\Pi_Q)$ that, when run on every $M \in \mathcal{M}$, correctly returns an optimal action $a_M^{\mu} \in \pi_M^*(s_0)$ with high probability. That is to say,

$$\max_{\mu \in \mathcal{P}(\Pi_Q)} \min_{M \in \mathcal{M}} P_{a \sim \mu(M)} \left(a_M^{\mu} \in \pi_M^*(s_0) \right) \ge 1 - \xi \,. \tag{12}$$

By Yao's minimax principle,

$$\min_{D \in \mathcal{P}(\mathcal{M})} \max_{\mu \in \Pi_Q} P_{M \sim D} \left(a_M^{\mu} \in \pi_M^*(s_0) \right) \ge 1 - \xi \,, \tag{13}$$

where D is a distribution on \mathcal{M} .

Let D_1 and D_2 be uniform distributions on \mathcal{M}_1 and \mathcal{M}_2 , respectively, and let D be their uniform mixture. Now let $\mu \in \Pi_Q$ be a deterministic algorithm which fails to return an optimal $a_M^{\mu} \in \pi_M^*(s_0)$ with a probability of at most ξ on inputs from D. This implies that μ fails with a probability of at most 2ξ if the instance is drawn from either of D_1 or D_2 considered individually. We define $\mathcal{C}_i \subset \mathcal{M}_i$ as the sets of instances on which μ succeeds. It is obvious that

$$|\mathcal{C}_i| \ge (1-2\xi)|\mathcal{M}_i| = (1-2\xi)mn^m$$

We call $M_1 \in \mathcal{M}_1$ and $M_2 \in \mathcal{M}_2$ a *twin* if their transition kernels are identical except that the reward for taking action \bar{a} at state \bar{s} is r_i for i = 1, 2. We let $E(\mathcal{A}_1, \mathcal{A}_2)$ denote the number of twins where the *i*-th component of the triplet is in \mathcal{A}_i for i = 1, 2. The number of twins on which μ succeeds is lower bounded by

$$E(\mathcal{C}_1, \mathcal{C}_2) \ge E(\mathcal{C}_1, \mathcal{M}_2) - E(\mathcal{M}_1, \mathcal{M}_2 \setminus \mathcal{C}_2)$$

$$\ge (1 - 2\xi) \cdot mn^m - 2\xi \cdot mn^m = (1 - 4\xi)mn^m.$$

Setting $\xi = \frac{1}{8}$ guarantees that μ distinguishes at least $\frac{1}{2}mn^m$ twins of DP instances. The key observation now is that for any twin, μ has to query (\bar{s}, \bar{a}) , that is, the special state–action pair associated to the twin; otherwise, μ cannot distinguish them. We now define a new problem.

Definition V.1 (Vector differentiation). Let A and B be two arrays of integers of size m. We say that a deterministic algorithm is able to distinguish A from B if it queries an entry i for which $A_i \neq B_i$. We say (A, B) is a twin of arrays of size m if A takes values in $\{1, 2, ..., n\}$ and B differs from A in only one entry. At this entry, B takes a value 0.

Now, we use μ to construct an algorithm which applies to the vector differentiation problem. We first construct an auxiliary algorithm $\tilde{\mu}$ that mimics the flow of μ , but with a slight difference. Suppose that μ examines an entry (s, a). If $a(s) \in S_G$, then $\tilde{\mu}$ will branch similarly to μ as though the query to a(s) has returned 0. As a result, $\tilde{\mu}$ branches exactly like μ on inputs from C_1 . This, consequently, means that on any component of any triplet that μ can distinguish, $\tilde{\mu}$ queries (\bar{s}, \bar{a}) but collects $\bar{a}(\bar{s})$ from an instance in C_1 and 0 from an instance in C_2 . Therefore, vector differentiation can be reduced to $\tilde{\mu}$. The following observation now remains to be made:

Proposition V.1. Any deterministic algorithm $\tilde{\mu}$ that performs vector differentiation needs $\Omega(m)$ queries to distinguish $\frac{1}{2}mn^m$ twins of vectors.

Proof. We view the queries of $\tilde{\mu}$ as a decision tree. At every node of the tree, $\tilde{\mu}$ queries a certain element of the vector of integers. The root of the tree is the beginning of the algorithm at which no queries have yet been made. We say this node is at depth 0. A node at which a k-th query to the vector is made is called a depth-k node. It is obvious that a depth-k node can distinguish at most n^{m-k} pairs of vectors. Let (A, B) be a twin, with A and B distinguishable at a depth-k node. This means that all previous k - 1 queries to A and B have returned in the same integers. The k-th query has resulted in a nonzero integer on one of the vectors and 0 on the other. There are m - k remaining entries and A and B have to coincide for all of them. This means that there are n^{m-k} ways to form A and B into twins.

On the other hand, there are at most n^k nodes at a depth of k. Therefore, the depth-k nodes can in total distinguish at most n^m twins of vectors. In order for $\tilde{\mu}$ to distinguish $\frac{1}{2}mn^m$ twin vectors, the total depth of the decision tree of $\tilde{\mu}$ has to be at least $\frac{1}{2}m$. This proves the claim.

Corollary 5. Any classical randomized algorithm that solves a dynamic programming problem at a marked initial state and a time horizon $T > \log(\lceil |S| \rceil)$ via queries to an oracle

$$(s,a) \mapsto \left(a(s), r(s,a)\right)$$

has to make at least $\Omega(|S||A|)$ queries to that oracle.

VI. Conclusion

In this paper, we have introduced and analyzed quantum algorithms for solving finite-horizon dynamic programming problems. Several complexity-theoretic arguments were used to demonstrate

a quantum advantage in terms of the size of the dynamic programming problem solved at the expense of the appearance of polynomial factors of parameters that represent the precision of the solution and the number of actions.

The precision factor for finite-horizon dynamic programming problems appeared as an upper bound on the instantaneous reward (or cost) of each action in the formulation of the problem. In contrast, the polynomial scaling with respect to the number of actions is required in order to guarantee that an optimal action can be read out from the approximate solution generated by the multiplicative weight update method.

As a first application, we used our algorithm to solve the travelling salesperson problem (TSP). Given a maximum edge-weight $\lceil c \rceil$, we showed that our algorithm can solve the TSP using $O^*(\lceil c \rceil^4 \sqrt{2^n})$ quantum gates, providing a quadratic speedup in solving the TSP at the expense of a polynomial (quartic) scaling with respect to the maximum edge-weight $\lceil c \rceil$. It is important to note that both the classical solution of Bellman–Held–Karp [Bel61, HK62] with a runtime of $O(n^22^n)$ and the quantum algorithm of $[ABI^+19]$ with a runtime of $O^*(1.728^n)$ have only polylog-arithmic dependence on the maximum edge-weight. Our result is perhaps better comparable to classical algorithms that take advantage of small edge-weights. For example, [Bjo14] presents an algorithm that has a runtime of $O^*(w1.657^n)$, in which a linear dependence on the sum w of all weights in the graph appears.

As a second application, we used our algorithm to solve the minimum set-cover problem (MSC). Given a universe with n elements, and m of its subsets, our quantum algorithm solves the problem of finding a minimal cover for the universe using $O(\sqrt{2^n} \operatorname{poly}(m, n))$ gates. We note that various classical algorithms for the MSC exist with running times $O(n2^m)$, $O(mn2^n)$ [FK10], and $O(1.227^{m+n})$ [VRB08]. The dynamic programming approach results in a run-time of $O(mn2^n)$.

One important direction for future research would be to improve the dependence of the approach presented herein on the number of actions and the precision parameters. It would be interesting to investigate whether polynomial dependence on these parameters is necessary. In particular, we raise the question of whether a quantum algorithm for solving the TSP that achieves $O^*(\sqrt{2^n})$ scaling in *n* would necessarily require a polynomial dependence on the maximum edge-weight of the graph.

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[[]Aar06] Scott Aaronson. Lower bounds for local search by quantum arguments. SIAM Journal on Computing, 35(4):804–824, 2006.

[[]ABI+19] Andris Ambainis, Kaspars Balodis, Jānis Iraids, Martins Kokainis, Krišjānis Prūsis, and Jevgēnijs Vihrovs. Quantum speedups for exponential-time dynamic programming algorithms. In Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, pages 1783–1793. SIAM, 2019.

- [AG18] Joran van Apeldoorn and András Gilyén. Improvements in quantum sdp-solving with applications. arXiv preprint arXiv:1804.05058, 2018.
- [AG19] Joran van Apeldoorn and András Gilyén. Quantum algorithms for zero-sum games. arXiv preprint arXiv:1904.03180, 2019.
- [AGGW17] Joran van Apeldoorn, András Gilyén, Sander Gribling, and Ronald de Wolf. Quantum sdpsolvers: Better upper and lower bounds. In Foundations of Computer Science (FOCS), 2017 IEEE 58th Annual Symposium on, pages 403–414. IEEE, 2017.
- [AK07] Sanjeev Arora and Satyen Kale. A combinatorial, primal-dual approach to semidefinite programs. In Proceedings of the thirty-ninth annual ACM symposium on Theory of computing, pages 227–236. ACM, 2007.
- [Amb02] Andris Ambainis. Quantum lower bounds by quantum arguments. Journal of Computer and System Sciences, 64(4):750–767, 2002.
- [BC12] Hans J Briegel and Gemma De las Cuevas. Projective simulation for artificial intelligence. Scientific reports, 2:400, 2012.
- [Bel61] Richard Ernest Bellman. Dynamic programming treatment of the traveling salesman problem. 1961.
- [Bel13] Richard Bellman. Dynamic programming. Courier Corporation, 2013.
- [Bjo14] Andreas Bjorklund. Determinant sums for undirected hamiltonicity. SIAM Journal on Computing, 43(1):280–299, 2014.
- [BS17] Fernando GSL Brandão and Krysta M Svore. Quantum speed-ups for solving semidefinite programs. In Foundations of Computer Science (FOCS), 2017 IEEE 58th Annual Symposium on, pages 415–426. IEEE, 2017.
- [CLG⁺18] Daniel Crawford, Anna Levit, Navid Ghadermarzy, Jaspreet S Oberoi, and Pooya Ronagh. Reinforcement learning using quantum boltzmann machines. *Quantum Information and Computation*, 18, January 2018.
- [CW17] Yichen Chen and Mengdi Wang. Lower bound on the computational complexity of discounted markov decision problems. arXiv preprint arXiv:1705.07312, 2017.
- [DCT⁺08] Daoyi Dong, Chunlin Chen, Tzyh-Jong Tarn, Alexander Pechen, and Herschel Rabitz. Incoherent control of quantum systems with wavefunction-controllable subspaces via quantum reinforcement learning. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, 38(4):957–962, 2008.
- [DH96] Christoph Durr and Peter Hoyer. A quantum algorithm for finding the minimum. arXiv preprint quant-ph/9607014, 1996.
- [FK10] Fedor V Fomin and Dieter Kratsch. Exact exponential algorithms. Springer Science & Business Media, 2010.
- [HK62] Michael Held and Richard M Karp. A dynamic programming approach to sequencing problems. Journal of the Society for Industrial and Applied Mathematics, 10(1):196–210, 1962.
- [Kal07] Satyen Kale. Efficient algorithms using the multiplicative weights update method. Princeton University, 2007.
- [KLL⁺19] Amir Kalev, Tongyang Li, Cedric Yen-Yu Lin, Krysta M Svore, Xiaodi Wu, et al. Quantum sdp solvers: Large speed-ups, optimality, and applications to quantum learning. *Leibniz international* proceedings in informatics, 2019.
- [MGAG16] Salvatore Mandra, Gian Giacomo Guerreschi, and Alán Aspuru-Guzik. Faster than classical quantum algorithm for dense formulas of exact satisfiability and occupation problems. *New Journal of Physics*, 18(7):073003, 2016.
- [MLM17] Dominic J Moylett, Noah Linden, and Ashley Montanaro. Quantum speedup of the travelingsalesman problem for bounded-degree graphs. *Physical Review A*, 95(3):032323, 2017.
- [VRB08] Johan MM Van Rooij and Hans L Bodlaender. Design by measure and conquer, a faster exact algorithm for dominating set. arXiv preprint arXiv:0802.2827, 2008.